Directed Reading Program Fall 2020
Saddle Point Theorem

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1 Preliminaries

Let $f_0 : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function, $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, ..., p$ be differentiable convex functions and $h_i : \mathbb{R}^n \to \mathbb{R}$ be affine functions for $i = 1, ..., q$. We consider the following **constrained minimization problem** or the **primal problem**

$$\inf_{x \in \Omega} f_0(x), \tag{1}$$

where

$$\Omega := \{x \in \mathbb{R}^n \mid f_i(x) \leq 0 \text{ for } i = 1, ..., p \text{ and } h_i(x) = 0 \text{ for } i = 1, ..., q\}. \tag{2}$$

The **Lagrangian** corresponding to the minimization problem (1) is defined as

$$L(x, y, z) = f_0(x) + \sum_{i=1}^{p} y_i f_i(x) + \sum_{i=1}^{q} z_i h_i(x), \tag{3}$$

where $y \in \mathbb{R}^p_+$ and $z \in \mathbb{R}^q$ are called **Lagrange multipliers**. We define the dual of the Lagrangian L to be

$$g(y, z) = \inf_{x \in \Omega} L(x, y, z). \tag{4}$$

Thus, the **dual problem** of (1) is given by

$$\sup_{(y,z) \in K} g(y, z), \tag{5}$$

where $K := \mathbb{R}^p_+ \times \mathbb{R}^q$. A point $(\overline{x}, \overline{y}, \overline{z}) \in \mathbb{R}^n \times K$ is said to be a **saddle point** for L if it satisfies

$$L(\overline{x}, y, z) \leq L(\overline{x}, \overline{y}, \overline{z}) \leq L(x, \overline{y}, \overline{z}) \tag{6}$$

where $(x, y, z) \in \mathbb{R}^n \times K$.

Definition 1.1 (Slater Constraint Qualification)**.** We say that Ω defined in (2) satisfies the *slater constraint qualification* if there exists $\tilde{x} \in \Omega$ such that $f_i(\tilde{x}) < 0$ for $i = 1, ..., p$.

We know that the Slater's condition implies the strong duality, i.e. there exist $\overline{x} \in \Omega$ and $(\overline{y}, \overline{z}) \in K$ such that $f_0(\overline{x}) = g(\overline{y}, \overline{z})$.

Definition 1.2 (Feasibility and Optimality)**.** We say that the solution x is a **feasible solution** of a primal problem if $x \in \Omega$. We say that the solution $\overline{x}$ of a primal problem (1) is an **optimal solution** if it is feasible and satisfies $f_0(\overline{x}) \leq f_0(x)$ for all $x \in \Omega$.

Next, we state without proof an important theorem for the optimality condition of a solution to the optimization problem. This is called **Karush-Kuhn-Tucker theorem** or **KKT condition**.

Theorem 1.3. *A triplet* $(\overline{x}, \overline{y}, \overline{z}) \in \mathbb{R}^n \times K$ *is said to be a* ***KKT triplet*** *if it satisfies the following* ***KKT conditions***

1. *(Primal feasibility)* $f_i(x) = 0$ *for* $i = 1, ..., p$ *and* $h_i(x) = 0$ *for* $i = 1, ..., q$.
2. *(Dual feasibility)* $y_i \geq 0$ *for* $i = 1, ..., p$.
3. *(Complementary slackness)* $y_i f_i(x) = 0$ *for* $i = 1, ..., p$.
4. $\nabla_x L(x, y, z) = 0$

If $(\overline{x}, \overline{y}, \overline{z}) \in \Omega \times K$ *is KKT triplet then* $\overline{x}$ *is an primal optimal and* $(\overline{y}, \overline{z})$ *is a dual optimal with zero duality gap.*

If the minimization problem (1) has a differentiable and convex cost function f_0 and also f_i are differentiable and convex, for $i = 1, ..., p$ satisfying the Slater's condition, then the KKT theorem gives the necessary and sufficient condition for optimality, i.e. $(\overline{x}, \overline{y}, \overline{z}) \in \Omega \times K$ is a KKT triplet if and only if $\overline{x}$ is an primal optimal and $(\overline{y}, \overline{z})$ is a dual optimal.

2 Saddle Point Theorem

Theorem 2.1 (Saddle Point Theorem)**.** *Let* $\overline{x} \in \mathbb{R}^n$*, if there exists* $(\overline{y}, \overline{z}) \in K$ *such that* $(\overline{x}, \overline{y}, \overline{z})$ *is a saddle point for the Lagrangian* L*, then* $\overline{x}$ *solve* (1)*. Conversely, if* $\overline{x}$ *is the optimal solution to* (1) *at which the Slater's condition holds, then there is* $(\overline{y}, \overline{z})$ *such that* $(\overline{x}, \overline{y}, \overline{z})$ *is a saddle point for* L*.*

Proof. To start with, we want to show that if there exists $(\overline{x}, \overline{y}, \overline{z}) \in R^n \times K$ such that $(\overline{x}, \overline{y}, \overline{z})$ satitisfies the saddle point condition (6) then $\overline{x}$ is the optimal solution to (1).

First we show that $\overline{x}$ is a feasible solution. Consider

$$L(\overline{x}, y, z) = f_0(\overline{x}) + \sum_{i=1}^{p} y_i f_i(\overline{x}) + \sum_{i=1}^{q} z_i h_i(\overline{x}),$$

where $y_i \geq 0$. Using the definition of a saddle point (6), we have that

$$\sup_{(y,z) \in K} L(\overline{x}, y, z) \leq L(\overline{x}, \overline{y}, \overline{z})$$

It is clear that $h_i(\overline{x})$ must be 0; otherwise, we choose z to be $\text{sgn}(h_i(\overline{x}))\infty$ and thus $\sup_{(y,z)\in K} L(\overline{x}, y, z) = +\infty$, which is absurd. Also, $f_i(\overline{x})$ must be less than or equal to 0; otherwise, we can let $y_i \to +\infty$, which gives $\sup_{(y,z)\in K} L(\overline{x}, y, z) = +\infty$. Hence, $\overline{x}$ is a feasible solution to (1). Now we want to show that $\overline{x}$ is the optimal solution to (1) i.e. we need to show $f_0(\overline{x}) \leq f_0(x)$ for all $x \in \Omega$. Using the saddle point condition (6), $L(\overline{x}, y, z) \leq L(\overline{x}, \overline{y}, \overline{z})$,

$$f_0(\overline{x}) + \sum_{i=1}^{p} y_i f_i(\overline{x}) + \sum_{i=1}^{q} z_i h_i(\overline{x}) \leq f_0(\overline{x}) + \sum_{i=1}^{p} \overline{y}_i f_i(\overline{x}) + \sum_{i=1}^{q} \overline{z}_i h_i(\overline{x})$$

Since $\overline{x}$ is the feasible solution, $h_i(\overline{x}) = 0$ for $i = 1, ..., q$. Then we have,

$$\sum_{i=1}^{p} (y_i - \overline{y}_i) f_i(\overline{x}) \leq 0 \text{ for all } y_i \in \mathbb{R}^p_+.$$

Thus, we let $y = 0$ and using the fact that $y_i \geq 0$ and $f_i(\overline{x}) \leq 0$ for $i = 1, ..., p$ to conclude that

$$\sum_{i=1}^{p} \overline{y} f_i(\overline{x}) = 0$$

Now consider $f_0(\overline{x}) = L(\overline{x}, \overline{y}, \overline{z}) \leq L(x, \overline{y}, \overline{z})$, we have $f_0(\overline{x}) \leq \inf_{x\in\Omega} L(x, \overline{y}, \overline{z})$. Thus,

$$f_0(\overline{x}) \leq \inf_{x\in\Omega} f_0(x) + \inf_{x\in\Omega} \sum_{i=1}^{p} \overline{y}_i f_i(x)$$

With $f_i(x) \leq 0$ and $\overline{y}_i \geq 0$ $i = 1, ..., p$, we have $f_0(\overline{x}) \leq f_0(x)$ for all $x \in \Omega$. Thus, $\overline{x}$ is the optimal solution to the primal problem (1)

Conversely, suppose that $\overline{x}$ is the optimal solution to primal problem (1) and the Slater's condition holds. Since $\overline{x}$ is the solution to the primal problem (1), by Slater's condition, there exists the dual optimal $(\overline{y}, \overline{z}) \in K$ such that $(\overline{x}, \overline{y}, \overline{z})$ is the KKT triplet.

First, we show that $L(\overline{x}, y, z) \geq L(\overline{x}, \overline{y}, \overline{z})$. Using that $L(\overline{x}, y, z) = f_0(\overline{x}) + \sum_{i=1}^{p} y_i f_i(\overline{x}) + \sum_{i=1}^{q} z_i h_i(\overline{x})$ and $\overline{x}$ is a feasible solution, we then obtain

$$L(\overline{x}, y, z) \leq f(\overline{x}) = L(\overline{x}, \overline{y}, \overline{z})$$

The right-hand side follows from the complimentary slackness in Theorem (1.3), i.e. $\overline{y}_i f_i(\overline{x}) = 0$ for $i = 1, ..., p$. Next, we show that $L(\overline{x}, \overline{y}, \overline{z}) \geq L(x, \overline{y}, \overline{z})$. Consider

$$L(x, \overline{y}, \overline{z}) = f_0(x) + \sum_{i=1}^{p} \overline{y}_i f_i(x) + \sum_{i=1}^{q} \overline{z}_i h_i(x)$$

Since $f_i(x)$ for $i = 1, .., p$ is a convex function and $h_i(x)$ for $i = 1, ..q$ is an affine function, using the first order characterization of a convex function, we have

$$\begin{aligned} f_i(x) &\geq f_i(\overline{x}) + \nabla f_i(\overline{x}) \cdot (x - \overline{x}) \quad \text{for } i = 1, ..., p \\ h_i(x) &= h_i(\overline{x}) + \nabla h_i(\overline{x}) \cdot (x - \overline{x}) \quad \text{for } i = 1, ..., q \end{aligned}$$

This implies

$$L(x,\overline{y},\overline{z}) \geq f_0(\overline{x})+\sum_{i=1}^{p}\overline{y}_i f_i(\overline{x})+\sum_{i=1}^{q}\overline{z}_i h_i(\overline{x})+\left(\nabla f_0(\overline{x}) + \sum_{i=1}^{p}\overline{y}_i \nabla(f_i(\overline{x}) \sum_{i=1}^{q}\overline{z}_i \nabla h_i(x)\right)\cdot(x-\overline{x})$$

Using the last KKT condition in Theorem (1.3), i.e. $\nabla_x L(\overline{x},\overline{y},\overline{z}) = 0$, we obtain

$$\begin{aligned} L(x,\overline{y},\overline{z}) &\geq f_0(\overline{x}) + \sum_{i=1}^{p}\overline{y}_i f_i(\overline{x}) + \sum_{i=1}^{q}\overline{z}_i h_i(\overline{x}) \\ &= L(\overline{x},\overline{y},\overline{z}), \end{aligned}$$

which completes the proof. □

References

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[3] J. Mitchell, Introduction to Optimization, *Lecture note,* Rensselaer Polytechnic Institute, Fall2018.